Applied Mathematics Letters 23 (2010) 1464–1467

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Extinction and permanence in a stochastic non-autonomous population system☆

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ARTICLE INFO

Article history:
Received 17 November 2009
Received in revised form 21 July 2010
Accepted 1 August 2010

Keywords:
Stochastic non-autonomous predator–prey system
Holling II functional response
Extinction
Persistence

ABSTRACT

A stochastic non-autonomous predator–prey system with Holling II functional response is investigated. Sufficient criteria for extinction and uniform weak persistence in the mean for each species are established. The acute persistence–extinction thresholds for each species are obtained in many cases.

1. Introduction

Extinction and persistence of a predator–prey model with Holling II functional response is one of the important topics in mathematical biology. There are many successful persistence–extinction results for the deterministic autonomous case. Taking into account the effect of time-evolving environments, and assuming that the stochastic fluctuations, which should not be neglected in many cases, will manifest mainly the growth rates of the prey population and the predator population, we shall study the model

$$(SM):\begin{cases} \mathrm{d}x = x\left[r_1(t) - a_{11}(t)x - \dfrac{a_{12}(t)y}{1+\theta(t)x}\right]\mathrm{d}t + \alpha_1(t)x\mathrm{d}B_1(t), \\ \mathrm{d}y = y\left[r_2(t) + \dfrac{a_{21}(t)x}{1+\theta(t)x} - a_{22}(t)y\right] + \alpha_2(t)y\mathrm{d}B_1(t), \end{cases}$$

where $r_i(t)$, $a_{ij}(t)$, $\alpha_i(t)$ and $\theta(t)$ are continuous bounded functions on $R_+ = [0, +\infty)$, $\inf_{t\in R_+}\theta(t) > 0$, $\inf_{t\in R_+} a_{ij}(t) > 0$ and $\sup_{t\in R_+} r_2(t) < 0$; $\alpha_i^2(t)$ stands for the intensity of the white noise $\mathrm{d}B_i(t)$, $i, j = 1, 2$.

Definition. 1. Population x is said to go to extinction if for any initial value $x(0) = x_0 > 0$, we have $\lim_{t\to+\infty} x(t; 0, x_0) = 0$.
2. Population x is said to be uniformly weakly persistent in the mean [1] if there are constants $\beta > 0$ and $M > 0$ such that for any initial value $x_0 > 0$, we have $M \geq \limsup_{t\to+\infty}\langle x(t; 0, x_0)\rangle \geq \beta$, where $\langle f(t)\rangle = t^{-1}\int_0^t f(s)\mathrm{d}s$.

If $f(t)$ is a continuous bounded function on $[0, +\infty)$, define

$$f^u = \sup_{t\in R_+} f(t), \qquad f^l = \inf_{t\in R_+} f(t), \qquad b_i(t) = r_i(t) - \alpha_i^2(t)/2, \qquad c(x(t)) = \left\langle b_2(t) + \frac{a_{21}(t)x(t)}{1+\theta(t)x(t)}\right\rangle.$$

☆ This work was supported by the National Natural Science Foundation of PR China (No. 10701020 and 10971022).
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doi:10.1016/j.aml.2010.08.012

Our main results are the following theorems for the arbitrary solution $(x(t), y(t)) = (x(t; 0, x_0), y(t; 0, y_0))$ of model (SM) with initial values $x_0 > 0$ and $y_0 > 0$:

Theorem 1. *If* $\limsup_{t\to+\infty}\langle b_1(t)\rangle < 0$, *then both prey population x and predator population y will go to extinction almost surely (a.s.).*

Theorem 2. *If* $\limsup_{t\to+\infty}\langle b_1(t)\rangle > 0$ *and* $\limsup_{t\to+\infty} c(\tilde{x}(t)) < 0$ *a.s., then x will be uniformly weakly persistent in the mean and y will go to extinction a.s., where* $\tilde{x}(t) = \tilde{x}(t; 0, \tilde{x}_0)$ *is a solution of*

$$\mathrm{d}x = x[r_1(t) - a_{11}(t)x]\mathrm{d}t + \alpha_1(t)x\mathrm{d}B_1(t). \tag{1}$$

Theorem 3. *If* $\limsup_{t\to+\infty}\langle b_1(t)\rangle > 0$, *and there exists a number* σ *such that*

$$\limsup_{t\to+\infty} c(\tilde{x}(t)) \geq \sigma > 0 \quad \textit{a.s.}, \tag{2}$$

then both x and y will be uniformly weakly persistent in the mean a.s.

2. Proof

Lemma 4. *For model (SM), then for any given initial value* $(x_0, y_0) \in R^2_+$, *there is a unique solution* $(x(t), y(t))$ *on* $t \geq 0$ *and the solution will remain in* R^2_+ *with probability* 1. *Moreover, if* $a^l_{ii} > 0$ $(i = 1, 2)$, *then*

$$\limsup_{t\to+\infty}[\ln x(t)/t] \leq 0, \qquad \limsup_{t\to+\infty}[\ln y(t)/t] \leq 0. \tag{3}$$

Suppose that $\tilde{x}(t), \bar{x}(t)$ *are two arbitrary solutions of Eq.* (1) *with initial values* $\tilde{x}_0 \in R_+, \bar{x}_0 \in R_+$ *respectively; then* $\lim_{t\to+\infty} |\tilde{x}(t) - \bar{x}(t)| = 0$.

Proof. Consider the equations

$$\mathrm{d}u = \left[b_1(t) - a_{11}(t)\mathrm{e}^{u(t)} - \frac{a_{12}(t)\mathrm{e}^{v(t)}}{1+\theta(t)\mathrm{e}^{u(t)}}\right]\mathrm{d}t + \alpha_1(t)\mathrm{d}B_1(t), \tag{4}$$

$$\mathrm{d}v = \left[b_2(t) - a_{22}(t)\mathrm{e}^{v(t)} + \frac{a_{21}(t)\mathrm{e}^{u(t)}}{1+\theta(t)\mathrm{e}^{u(t)}}\right]\mathrm{d}t + \alpha_2(t)\mathrm{d}B_2(t) \tag{5}$$

on $t \geq 0$ with initial value $u_0 = \ln x_0, v_0 = \ln y_0$. Clearly, the coefficients of (4) and (5) satisfy the local Lipschitz condition; then there is a unique local solution $u(t), v(t)$ on $[0, \tau_e)$, where τ_e is the explosion time. Therefore, by Itô's formula, $x(t) = \mathrm{e}^{u(t)}, y(t) = \mathrm{e}^{v(t)}$ is the unique positive local solution to (SM) with initial value $x_0 > 0, y_0 > 0$. Now, let us show that this solution is global, i.e., $\tau_e = \infty$. The proof is a modification of [2, Theorem 2.1], by defining $V(x, y) = (x - 1 - \ln x) + (y - 1 - \ln y)$, and hence is omitted. To prove (3), note that (see e.g. [2, Lemma 3.4]) the solution of (1), $\tilde{x}(t)$, satisfies $\limsup_{t\to+\infty}[\ln \tilde{x}(t)/\ln t] \leq 1$. Then the desired assertion (3) follows from the comparison theorem for stochastic differential equations (SDEs). The last assertion follows from [2, Corollary 5]. □

Proof of Theorem 1. Applying Itô's formula to model (SM) gives

$$\frac{\ln(x(t)/x(0))}{t} = \langle b_1(t)\rangle - \langle a_{11}(t)x(t)\rangle - \left\langle \frac{a_{12}(t)y(t)}{1+\theta(t)x(t)}\right\rangle + \frac{U_1(t)}{t}, \tag{6}$$

$$\frac{\ln(y(t)/y(0))}{t} = \langle b_2(t)\rangle - \langle a_{22}(t)y(t)\rangle + \left\langle \frac{a_{21}(t)x(t)}{1+\theta(t)x(t)}\right\rangle + \frac{U_2(t)}{t} \tag{7}$$

where $U_i(t) = \int_0^t \alpha_i(t)\mathrm{d}B_i(t)$. By the strong law of large numbers for martingales, we can derive that

$$\lim_{t\to+\infty} U_i(t)/t = 0, \quad \text{a.s.} \tag{8}$$

In view of [1], we then obtain

$$\limsup_{t\to+\infty}\langle x(t)\rangle \leq |b^u_1|/a^l_{11}, \qquad \limsup_{t\to+\infty}\langle y(t)\rangle \leq |b^u_2 + a^u_{21}/\theta^l|/a^l_{22}.$$

By virtue of (6) and (8), $\limsup_{t\to+\infty} \ln x(t)/t \leq \limsup_{t\to+\infty}\langle b_1(t)\rangle < 0$. In other words, $\lim_{t\to+\infty} x(t) = 0$. Then it follows from (7) and (8) that $\limsup_{t\to+\infty} t^{-1}\ln y(t) \leq \limsup_{t\to+\infty}\langle b_2(t)\rangle < 0$.

Proof of Theorem 2. To begin with, we shall show that there exists a constant $\beta > 0$ such that for any solution $(x(t), y(t))$ of model (SM) with initial values $x_0 > 0$ and $y_0 > 0$, we have $\limsup_{t\to+\infty}\langle x(t)\rangle \geq \beta$ a.s. Otherwise, for arbitrary $\varepsilon > 0$, there exists a solution $(\hat{x}(t), \hat{y}(t))$ with positive initial values $\hat{x}_0 > 0$ and $\hat{y}_0 > 0$ such that $\mathcal{P}\{\limsup_{t\to+\infty}\langle\hat{x}(t)\rangle < \varepsilon\} > 0$. Let ε be sufficiently small that

$$\limsup_{t\to+\infty}\langle b_1(t)\rangle - a_{11}^u\varepsilon > 0, \qquad \limsup_{t\to+\infty}\langle b_2(t)\rangle + a_{21}^u\varepsilon/\theta^l < 0. \tag{9}$$

It then follows from (7)–(9) that $\limsup_{t\to+\infty}\ln\hat{y}(t)/t \le \limsup_{t\to+\infty}\langle b_2(t)\rangle + a_{21}^u\varepsilon/\theta^l < 0$. Thus $\lim_{t\to+\infty}\hat{y}(t) = 0$. On the other hand, by virtue of (6),

$$\ln(\hat{x}(t)/\hat{x}(0))/t \ge \langle b_1(t)\rangle - a_{11}^u\langle\hat{x}(t)\rangle - a_{12}^u\langle\hat{y}(t)\rangle + U_1(t)/t.$$

Taking the superior limit to the above inequality and making use of (8), (9) and $\lim_{t\to+\infty}\hat{y}(t) = 0$, we have $\limsup_{t\to+\infty} t^{-1}\ln\hat{x}(t) \ge \limsup_{t\to+\infty}\langle b_1(t)\rangle - a_{11}^u\varepsilon > 0$. In other words, we have shown that $\mathcal{P}\{\limsup_{t\to+\infty} t^{-1}\ln\hat{x}(t) > 0\} > 0$. This is a contradiction (see (3) in Lemma 4).

In order to show that the predator population will go to extinction, let $\bar{x}(t) = \bar{x}(t; 0, x_0)$ be a solution of (1) with initial value x_0. Using the comparison theorem for SDEs yields $x(t) \le \bar{x}(t)$, or

$$x(t)/(1+\theta(t)x(t)) \le \bar{x}(t)/(1+\theta(t)\bar{x}(t)),$$

where $x(t)$ is the solution of the prey population equation. Substituting the above inequality into (7) leads to

$$\begin{aligned}\frac{\ln(y(t)/y(0))}{t} &\le c(\tilde{x}(t)) + \left\langle a_{21}(t)\left(\frac{\bar{x}}{1+\theta(t)\bar{x}} - \frac{\tilde{x}}{1+\theta(t)\tilde{x}}\right)\right\rangle + \frac{U_2(t)}{t}\\ &\le c(\tilde{x}(t)) + a_{21}^u\langle|\tilde{x}(t) - \bar{x}(t)|\rangle + U_2(t)/t.\end{aligned}$$

Taking the superior limit to the above inequality and making use of the last assertion in Lemma 4 and (8), we can see that $\limsup_{t\to+\infty} t^{-1}\ln y(t) < 0$.

Proof of Theorem 3. From Theorem 2, we only need to show that $\limsup_{t\to+\infty}\langle y(t)\rangle \ge \beta$ a.s. Otherwise, for arbitrary fixed $\varepsilon > 0$, there exists a solution $(\check{x}(t), \check{y}(t))$ of (SM) with positive initial values $\check{x}_0 > 0$ and $\check{y}_0 > 0$ such that $\mathcal{P}\{\limsup_{t\to+\infty}\langle\check{y}(t)\rangle < \varepsilon\} > 0$. Let ε be sufficiently small that $\sigma - 2a_{12}^u a_{21}^u\varepsilon/a_{11}^l - a_{22}^u\varepsilon > 0$. It follows from (7) that

$$t^{-1}\ln(\check{y}(t)/\check{y}(0)) \ge c(\tilde{x}(t)) - a_{21}^u\langle|\tilde{x}(t) - \check{x}(t)|\rangle - \langle a_{22}(t)\check{y}(t)\rangle + U_2(t)/t. \tag{10}$$

On the other hand, consider a Lyapunov function $V(t) = |\ln\check{x}(t) - \ln\tilde{x}(t)|$. A direct calculation of the right differential $d^+V(t)$ results in

$$d^+V(t) \le [a_{12}^u\check{y}(t)/(1+\theta(t)\check{x}(t)) - a_{11}^l|\tilde{x}(t) - \check{x}(t)|]\mathrm{d}t,$$

which indicates that $\langle|\tilde{x}(t) - \check{x}(t)|\rangle \le a_{12}^u\varepsilon/a_{11}^l + V(0)/(ta_{11}^l)$. We can choose t large enough that $V(0)/t \le a_{12}^u\varepsilon$. Then $\langle|\tilde{x}(t) - \check{x}(t)|\rangle \le 2a_{12}^u\varepsilon/a_{11}^l$. Substituting the above inequality into (10) and taking the superior limit yields

$$\begin{aligned}\limsup_{t\to+\infty} t^{-1}\ln(\check{y}(t)) &\ge \limsup_{t\to+\infty} c(\tilde{x}(t)) - 2a_{12}^u a_{21}^u\varepsilon/a_{11}^l - a_{22}^u\varepsilon\\ &\ge \sigma - 2a_{12}^u a_{21}^u\varepsilon/a_{11}^l - a_{22}^u\varepsilon > 0,\end{aligned}$$

which implies that $\mathcal{P}\{\limsup_{t\to+\infty}\ln\check{y}(t)/t > 0\} > 0$. This is a contradiction.

Remark 1. If $\alpha_1 = 0$, then condition (2) can be changed to $\limsup_{t\to+\infty} c(\tilde{x}(t)) > 0$. That is to say, the threshold between uniform weak persistence in the mean and extinction of both prey population and predator population is obtained. In particular, if $r_1(t) \equiv r_1$ and $a_{11}(t) \equiv a_{11}$, then the representations $\limsup_{t\to+\infty} c(\tilde{x}(t))$ and $\limsup_{t\to+\infty} c(\tilde{x}(t)) > \sigma$ in Theorems 2 and 3 respectively can be replaced by $\limsup_{t\to+\infty}\langle b_2(t) + \frac{a_{21}(t)r_1/a_{11}}{1+\theta(t)r_1/a_{11}}\rangle$.

3. Results and discussion

This work studied a stochastic non-autonomous predator–prey system with Holling II functional response. We established the sufficient conditions for extinction and uniform weak persistence in the mean and obtained the acute threshold in many cases. Owing to its theoretical and practical significance, the predator–prey system with Holling II functional response has attracted a lot of attention, but mainly for the deterministic case. The present work is the first attempt, to our knowledge, to carry out such a study in a stochastic setting.

Some interesting topics deserve further investigation. One may propose some more realistic but complex models. An example is the fractional predator–prey model. The motivation for studying this is that in the real world, discontinuity is a common phenomenon and many real objects or phenomena are generally discontinuous, and the most important thing concerning the population model is its discontinuous characteristic in time. It has been noted that (see e.g. [3]) fractional calculus is valid for discontinuous problems. Owing to its theoretical and practical significance, in recent years, the fractional effect has received great attention and has been studied extensively. In particular, Das and co-workers gave many important predictions for different fractional orders; see e.g. [4–6]; also, He et al. [3] pointed out the physical understanding of the fractional equations and proposed a fractional predator–prey model (equations (77)–(78)). These studies are very good references in this area.

Acknowledgements

The authors thank the referees very much for their very important and helpful comments and suggestions. The authors thank Dr. Subir Das and Dr. Francis Austin for their enthusiastic assistance.

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